Global existence and optimal decay of solutions to the dissipative Timoshenko system (Mathematical Analysis of Viscous Incompressible Fluid)

森, 直文; 川島, 秀一

数理解析研究所講究録
第 1971 巻 2015 年 150-164

Global existence and optimal decay of solutions to the dissipative Timoshenko system

Naofumi Mori
Graduate School of Mathematics, Kyushu University

Shuichi Kawashima
Faculty of Mathematics, Kyushu University

1 Introduction

In this paper we consider the nonlinear version of the dissipative Timoshenko system

$$\begin{cases} \varphi_{tt}-(\varphi_{x}-\psi)_{x}=0, \\ \psi_{tt}-\sigma(\psi_{x})_{x}-(\varphi_{x}-\psi)+\gamma\psi_{t}=0 \end{cases} \tag{1.1}$$

with the initial data

$$(\varphi, \varphi_{t}, \psi, \psi_{t})(x, 0)=(\varphi_{0}, \varphi_{1}, \psi_{0}, \psi_{1})(x)$$

in the one-dimensional whole space. The original Timoshenko system ($\gamma=0$) was first introduced by S.P. Timoshenko in [6, 7] as a model system which describes the vibration of the beam called Timoshenko beam: It considers not only transversal movement but also shearing deformation. Here $t\geq 0$ is a time variable and $x\in \mathbb{R}$ is a spacial variable which denotes a point on the center line of the beam; φ and ψ are unknown functions of (x, t), which denote the transversal displacement and the rotation angle of the beam, respectively. Note that $\varphi_{x}-\psi$ denotes the shearing stress. The nonlinear term $\sigma(\eta)$ is assumed to be a smooth function of η such that $\sigma'(\eta)>0$ for any η under considerations; the coefficient γ in the frictional damping term $\gamma\psi_{t}$ is a positive constant. The Timoshenko system is very important as a model system of symmetric hyperbolic systems or symmetric hyperbolic-parabolic systems because the system has new dissipative structures which can not be characterized by the general theory established by S. Kawashima in [5, 8] in terms of *the Kawashima condition*. In this paper we investigate the nonlinear version of the system by introducing frictional damping as the dissipative mechanism, and first prove the global existence and uniqueness of solutions under smallness assumption on the initial data in the Sobolev space H^{2} (with the critical regularity-index). Also, for initial data in $H^{2}\cap L^{1}$, we show that the solutions decay in L^{2} at the the optimal rate $t^{-1/4}$ for $t\rightarrow\infty$.

1.1 Formulation of the problem

By introducing new unknown functions

$$v := \varphi_x - \psi, \quad u := \varphi_t, \quad z := a\psi_x, \quad y := \psi_t,$$

we transform our system (1.1) into the first order hyperbolic system

$$v_t - u_x + y = 0, \tag{1.2a}$$

$$y_t - \sigma(z/a)_x - v + \gamma y = 0, \tag{1.2b}$$

$$u_t - v_x = 0, \tag{1.2c}$$

$$z_t - ay_x = 0, \tag{1.2d}$$

where $a := \sqrt{\sigma'(0)}$. The corresponding initial data are given by

$$(v, y, u, z)(x, 0) = (v_0, y_0, u_0, z_0)(x), \tag{1.3}$$

where $v_0 := \varphi_{0,x} - \psi_0$, $y_0 := \psi_1$, $u_0 := \varphi_1$, $z_0 := a\psi_{0,x}$. Here we remark that the nonlinearity of the system (1.2) depends on the component z only.

Our system (1.2) is a symmetric hyperbolic system with non-symmetric relaxation. In fact, we can write (1.2) as

$$A^0(z)W_t + A(z)W_x + LW = 0, \tag{1.4}$$

where $W = (v, y, u, z)^T$, $A^0(z) = \mathrm{diag}(1, 1, 1, b(z)/a)$ with $b(z) = \sigma'(z/a)/a$, and

$$A(z) = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b(z) \\ 1 & 0 & 0 & 0 \\ 0 & b(z) & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding linearized system at $z = 0$ is given by

$$W_t + AW_x + LW = 0, \tag{1.5}$$

where $A^0(0) = I$ and

$$A := A(0) = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}.$$

Notice that the linearized system is given explicitly as

$$\begin{cases} v_t - u_x + y = 0, \\ y_t - az_x - v + \gamma y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0. \end{cases} \tag{1.6}$$

In our system (1.4) or (1.5) the relaxation matrix L is not symmetric such that $\ker L \neq \ker L_1$, where L_1 denotes the symmetric part of L. This is the reason why the general theory on the dissipative structure developed in [5, 8] can not be applicable to our system.

1.2 Known results for linear system

The decay property of the linear system (1.6) was first investigated by J.E. Muñoz Rivera and R. Racke in [4]. They studied (1.6) in a bounded region and with simple boundary conditions and showed that the energy of the solution decays exponentially when $a = 1$, but polynomially when $a \neq 1$ as $t \to \infty$.

To explain this interesting decay property, K. Ide, K. Haramoto and S. Kawashima [2] considered the system (1.6) in one-dimensional whole space and showed that the dissipative structure of the system (1.6) can be described as

$$\operatorname{Re}\lambda(i\xi) \leq -c\eta(\xi), \qquad \eta(\xi) = \begin{cases} \xi^2/(1+\xi^2) & \text{for } a = 1, \\ \xi^2/(1+\xi^2)^2 & \text{for } a \neq 1, \end{cases}$$

where $\lambda(i\xi)$ denotes the eigenvalues of the system (1.6) in the Fourier space, and c is a positive constant. We note that the dissipative structure for $a = 1$ is the same as that in the general theory developed in [5, 8]. However, the dissipative strudture for $a \neq 1$ is much weaker in the high frequency region and causes regularity-loss in the dissipation term of the energy estimate and also in the decay estimate.

In fact, by using the energy method in the Fourier space, the authors in [2] derived the following pointwise estimate for the linear solution $W = (v, y, u, z)^T$ of (1.6):

$$|\hat{W}(\xi, t)| \leq Ce^{-c\eta(\xi)t}|\hat{W}_0(\xi)|,$$

where $W_0 = (v_0, y_0, u_0, z_0)^T$ is the corresponding initial data. Moreover, based on this pointwise estimate, they showed the optimal time decay estimates of the linear solution:

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-1/4-k/2}\|W_0\|_{L^1} + \begin{cases} Ce^{-ct}\|\partial_x^k W_0\|_{L^2} & \text{for } a = 1, \\ C(1+t)^{-\ell/2}\|\partial_x^{k+\ell} W_0\|_{L^2} & \text{for } a \neq 1, \end{cases}$$

where k and l are nonnegative integers, and C and c are positive constants. We note that when $a \neq 1$, in order to obtain the optimal decay rate $(1+t)^{-1/4-k/2}$ we have to assume the additional ℓ-th order regularity on the initial data to make the decay rate $(1+t)^{-\ell/2}$ better than $(1+t)^{-1/4-k/2}$. Therefore the regularity-loss can not be avoided in the decay estimate for $a \neq 1$.

1.3 Known results for nonlinear system

Based on these linear results in [2], K. Ide and S. Kawashima [1] proved the global existence and decay of solutions to the nonlinear system (1.2). To state the result, we introduce the following time-weighted norms $\tilde{E}(t)$ and $\tilde{D}(t)$:

$$\tilde{E}(t)^2 := \sum_{j=0}^{[s/2]} \sup_{0\leq\tau\leq t} (1+\tau)^{j-1/2}\|\partial_x^j W(\tau)\|^2_{H^{s-2j}},$$

$$\tilde{D}(t)^2 := \sum_{j=0}^{[s/2]} \int_0^t (1+\tau)^{j-3/2}\|\partial_x^j W(\tau)\|^2_{H^{s-2j}}\, d\tau$$

$$+ \sum_{j=0}^{[s/2]-1} \int_0^t (1+\tau)^{j-1/2}\|\partial_x^j v(\tau)\|^2_{H^{s-1-2j}}\, d\tau + \sum_{j=0}^{[s/2]} \int_0^t (1+\tau)^{j-1/2}\|\partial_x^j y(\tau)\|^2_{H^{s-2j}}\, d\tau.$$

Then the result in [1] is stated as follows.

Theorem 1.1 ([1]). *Assume that the initial data satisfy* $W_0 \in H^s \cap L^1$ *for* $s \geq 6$ *and put* $\tilde{E}_1 := \|W_0\|_{H^s} + \|W_0\|_{L^1}$. *Then there exists a positive constant* $\tilde{\delta}_1$ *such that if* $\tilde{E}_1 \leq \tilde{\delta}_1$, *the Cauchy problem* (1.2), (1.3) *has a unique global solution* $W(t)$ *with* $W \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$. *Moreover the solution* $W(t)$ *verifies the energy estimate*

$$\tilde{E}(t)^2 + \tilde{D}(t)^2 \leq C\tilde{E}_1^2$$

and the following optimal decay estimate for lower order derivatives

$$\|\partial_x^k W(t)\|_{L^2} \leq C\tilde{E}_1(1+t)^{-1/4-k/2},$$

where $0 \leq k \leq [s/2] - 1$, *and* $C > 0$ *is a constant.*

Remark 1.1. The result in Theorem 1.1 requires the regularity $s \geq 6$ and L^1 property on the initial data. Also, the norms $\tilde{E}(t)$ and $\tilde{D}(t)$ contains the time weights with negative exponents. These were crucial in [1] to overcome the difficulty caused by the regularity-loss property.

1.4 Main results

The main purpose of this paper is to refine Theorem 1.1 under less regularity assumption on the initial data. To state our results, we introduce the energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ by

$$E(t)^2 := \sup_{0\leq\tau\leq t} \|W(\tau)\|_{H^s}^2,$$

$$D(t)^2 := \int_0^t \|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^s}^2 + \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|\partial_x z(\tau)\|_{H^{s-1}}^2 \, d\tau.$$

Notice that in the dissipation norm $D(t)$ we have 1 regularity-loss for (v,u) but no regularity-loss for (y,z). Our first result is then stated as follows.

Theorem 1.2 (Global existence). *Assume that the initial data satisfy* $W_0 \in H^s$ *for* $s \geq 2$ *and put* $E_0 := \|W_0\|_{H^s}$. *Then there exists a positive constant* δ_0 *such that if* $E_0 \leq \delta_0$, *the Cauchy problem* (1.2), (1.3) *has a unique global solution* $W(t)$ *with* $W \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$. *Moreover the solution* $W(t)$ *verifies the energy estimate*

$$E(t)^2 + D(t)^2 \leq CE_0^2,$$

where $C > 0$ *is a constant.*

Remark 1.2. Our global existence result holds true under less regularity assumption $s \geq 2$ and without L^1 property on the initial data. This refinement is based on the better Lyapunov function constructed in [3]. Our Lyapunov function produces the optimal dissipation estimate for z without any regularity-loss (see $D(t)$), which enables us to control the nonlinearity depending only on z.

Next we state the result on the optimal time decay estimate.

Theorem 1.3 (Optimal L^2 decay estimate). *Assume that the initial data satisfy $W_0 \in H^2 \cap L^1$ and put $E_1 := \|W_0\|_{H^2} + \|W_0\|_{L^1}$. Then there is a positive constant δ_1 such that if $E_1 \leq \delta_1$, then the solution $W(t)$ obtained in Theorem 1.2 satisfies the following optimal L^2 decay estimate:*

$$\|W(t)\|_{L^2} \leq CE_1(1+t)^{-1/4},$$

where $C > 0$ is a constant.

Remark 1.3. In order to show the above decay estimate, we first estimate the nonlinear solution by using the energy method in the Fourier space and then apply the refined decay estimate of L^p-L^q-L^r type which was established in [9]. For the details, see Section 3.

Notations. Let $\hat{f} = \mathcal{F}[f]$ be the Fourier transform of f:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}} f(x)e^{-i\xi x}dx.$$

For $1 \leq p \leq \infty$, we denote by $L^p = L^p(\mathbb{R})$ the usual Lebesgue space on $\mathbb{R}$ with the norm $\|\cdot\|_{L^p}$. Also, for nonnegative integer s, we denote by $H^s = H^s(\mathbb{R})$ the Sobolev space of L^2 functions, equipped with the norm $\|\cdot\|_{H^s}$. In this paper, every positive constant is denoted by the same symbol C or c without confusion.

2 Energy method

The aim of this section is to prove the global existence result in Theorem 1.2. Our global existence result can be shown by the combination of a local existence result and the desired a priori estimate. Since our system (1.2) is a symmetric hyperbolic system, it is not difficult to show a local existence result by the standard method, and we omit the details. To state our result on the a priori estimate, we consider a solution $W(t)$ of the problem (1.2), (1.3) satisfying $W \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ for $s \geq 2$ and

$$\sup_{0 \leq t \leq T} \|W(t)\|_{L^\infty} \leq \delta, \tag{2.1}$$

where δ is a fixed positive constant. Our a priori estimate is now given as follows.

Proposition 2.1 (A priori estimate). *Suppose that $W_0 \in H^s$ for $s \geq 2$ and put $E_0 = \|W_0\|_{H^s}$. Let $T > 0$ and let $W(t)$ be a solution to the Cauchy problem* (1.2), (1.3) *satisfying* (2.1)*. Then there exists a positive constant δ_2 independent of T such that if $E_0 \leq \delta_2$, we have the a priori estimate*

$$E(t)^2 + D(t)^2 \leq CE_0^2, \qquad t \in [0,T], \tag{2.2}$$

where $C > 0$ is a constant independent of T.

To prove the above a priori estimate in Proposition 2.1, we need to show the following energy inequality by applying the energy method.

Proposition 2.2 (Energy inequality). *Suppose that $W_0 \in H^s$ for $s \geq 2$ and put $E_0 = \|W_0\|_{H^s}$. Let $T > 0$ and let $W(t)$ be a solution to the Cauchy problem* (1.2), (1.3) *satisfying* (2.1). *Then we have the following energy inequality:*

$$E(t)^2 + D(t)^2 \leq CE_0^2 + CE(t)D(t)^2, \qquad t \in [0, T], \tag{2.3}$$

where $C > 0$ is a constant independent of T.

We note that the desired a priori estimate (2.2) easily follows from the energy inequality (2.3), provided that E_0 is suitably small. Therefore it is sufficient to prove (2.3) for our purpose.

2.1 Proof of Proposition 2.2

In this subsection we prove the energy inequality (2.3) in Proposition 2.2 by using the energy method. Our energy method is based on the refined Lyapunov function constructed in [3] and gives the optimal dissipation estimate for z without any regularity-loss, which can control the nonlinearity of the system (1.2). Our proof is divided into 4 steps.

Step 1: (Basic energy and dissipation for y) We calculate as $(1.2a) \times v + (1.2b) \times y + (1.2c) \times u + (1.2d) \times \{\sigma(z/a) - \sigma(0)\}/a$. This yields

$$\frac{1}{2}(v^2 + y^2 + u^2 + S(z))_t - \{vu + (\sigma(z/a) - \sigma(0))y\}_x + \gamma y^2 = 0, \tag{2.4}$$

where $S(z) := 2\int_0^{z/a}(\sigma(\eta) - \sigma(0))d\eta$ is equivalent to $|z|^2$. Integrate (2.4) with respect to x to have

$$\frac{d}{dt}E_0^{(0)} + 2\gamma\|y\|_{L^2}^2 = 0, \tag{2.5}$$

where

$$E_0^{(0)} := \|(v, y, u)\|_{L^2}^2 + \int_{\mathbb{R}} S(z)dx.$$

Since $E_0^{(0)}$ is equivalent to $\|W\|_{L^2}^2$, by integrating (2.5) with respect to t, we obtain

$$\|W(t)\|_{L^2}^2 + \int_0^t \|y(\tau)\|_{L^2}^2 d\tau \leq CE_0^2. \tag{2.6}$$

Next, we apply ∂_x^k to (1.2) and write $\partial_x^k(v, y, u, z) = (V, Y, U, Z)$ for simplicity. Then we have

$$V_t - U_x - Y = 0, \tag{2.7a}$$

$$Y_t - \sigma'(z/a)(Z/a)_x - V + \gamma Y = [\partial_x^k, \sigma'(z/a)](z/a)_x, \tag{2.7b}$$

$$U_t - V_x = 0, \tag{2.7c}$$

$$Z_t - aY_x = 0, \tag{2.7d}$$

where $[A, B] := AB - BA$. We compute as $(2.14) \times V + (2.7b) \times Y + (2.17) \times U + (2.20) \times \sigma'(z/a)Z/a^2$. This gives

$$\frac{1}{2}\left(V^2 + Y^2 + U^2 + \sigma'(z/a)(Z/a)^2\right)_t - \{VU + \sigma'(z/a)(Z/a)Y\}_x + \gamma Y^2$$
$$= \frac{1}{2}\sigma'(z/a)_t(Z/a)^2 - \sigma'(z/a)_x(Z/a)Y + Y[\partial_x^k, \sigma'(z/a)](z/a)_x. \tag{2.8}$$

Integrate (2.8) with respect to x to have

$$\frac{d}{dt}E_0^{(k)} + 2\gamma\|\partial_x^k y\|_{L^2}^2 \le CR_0^{(k)} \tag{2.9}$$

for $1 \le k \le s$, where

$$E_0^{(k)} := \|\partial_x^k(v, y, u)\|_{L^2}^2 + \int_{\mathbb{R}} \sigma'(z/a)|\partial_x^k(z/a)|^2 dx,$$
$$R_0^{(k)} := \int_{\mathbb{R}} |y_x||\partial_x^k z|^2 + |z_x||\partial_x^k z||\partial_x^k y| + |[\partial_x^k, \sigma'(z/a)]z_x||\partial_x^k y| dx.$$

Here in the term $R_0^{(k)}$ we used the relation $z_t = ay_x$ from (1.2d). Now we integrate (2.9) with respect to t and add for k with $1 \le k \le s$. Since $E_0^{(k)}$ is equivalent to $\|\partial_x^k W\|_{L^2}^2$, we obtain

$$\|\partial_x W(t)\|_{H^{s-1}}^2 + \int_0^t \|\partial_x y(\tau)\|_{H^{s-1}}^2 d\tau \le CE_0^2 + CE(t)D(t)^2. \tag{2.10}$$

Here we have used the following estimates for $R_0^{(k)}$:

$$R_0^{(k)} \le C\|\partial_x(y, z)\|_{L^\infty}\|\partial_x^k(y, z)\|_{L^2}^2, \qquad \sum_{k=1}^{s}\int_0^t R_0^{(k)}(\tau)d\tau \le CE(t)D(t)^2.$$

Consequently, adding (2.6) and (2.10), we arrive at

$$E(t)^2 + \int_0^t \|y(\tau)\|_{H^s}^2 d\tau \le CE_0^2 + CE(t)D(t)^2. \tag{2.11}$$

Step 2: (Dissipation for v) We rewrite the system (1.2) in the form

$$\begin{aligned} &v_t - u_x - y = 0, \\ &y_t - az_x - v + \gamma y = g(z)_x, \\ &u_t - v_x = 0, \\ &z_t - ay_x = 0, \end{aligned} \tag{2.12}$$

where $g(z) := \sigma(z/a) - \sigma(0) - \sigma'(0)z/a = O(z^2)$ as $z \to 0$. We apply ∂_x^k to (2.12). Letting $(V, Y, U, Z) = \partial_x^k(v, y, u, z)$ as before, we have

$$V_t - U_x - Y = 0, \tag{2.13a}$$
$$Y_t - aZ_x - V + \gamma Y = \partial_x^k g(z)_x, \tag{2.13b}$$
$$U_t - V_x = 0, \tag{2.13c}$$
$$Z_t - aY_x = 0. \tag{2.13d}$$

To create the dissipation term V^2, we compute as (2.13b) $\times (-V)$ + (2.13a) $\times (-Y)$ + (2.13c) $\times (-aZ)$ + (2.13d) $\times (-aU)$. This gives

$$\begin{aligned} & -(VY + aUZ)_t + (aVZ + a^2YU)_x + V^2 \\ & = Y^2 + \gamma VY + (a^2-1)\,YU_x - V\partial_x^k g(z)_x. \end{aligned} \tag{2.14}$$

Integrate (2.14) with respect to x to obtain

$$\begin{aligned} \frac{d}{dt}E_1^{(k)} + \|\partial_x^k v\|_{L^2}^2 \le \|\partial_x^k y\|_{L^2}^2 + \gamma\|\partial_x^k v\|_{L^2}\|\partial_x^k y\|_{L^2} \\ + (a^2-1)\int_{\mathbb{R}} \partial_x^k y\,\partial_x^k u_x\,dx + R_1^{(k)} \end{aligned} \tag{2.15}$$

for $0 \le k \le s-1$, where

$$\begin{aligned} E_1^{(k)} &:= -\int_{\mathbb{R}} \partial_x^k v\,\partial_x^k y\,dx - a\int_{\mathbb{R}} \partial_x^k u\,\partial_x^k z\,dx, \\ R_1^{(k)} &:= \int_{\mathbb{R}} |\partial_x^k v|\,|\partial_x^{k+1} g(z)|\,dx. \end{aligned}$$

Adding (2.15) with k and $k+1$ and integrating by parts, we have

$$\begin{aligned} \frac{d}{dt}&(E_1^{(k)} + E_1^{(k+1)}) + \|\partial_x^k v\|_{H^1}^2 \le \|\partial_x^k y\|_{H^1}^2 + \gamma\|\partial_x^k v\|_{H^1}\|\partial_x^k y\|_{H^1} \\ &+ (a^2-1)\int_{\mathbb{R}} (\partial_x^k y\,\partial_x^k u_x - \partial_x^{k+1} y_x\,\partial_x^{k+1} u)\,dx + R_1^{(k)} + R_1^{(k+1)} \\ \le{}& \|\partial_x^k y\|_{H^1}^2 + \gamma\|\partial_x^k v\|_{H^1}\|\partial_x^k y\|_{H^1} + |a^2-1|\|\partial_x^k y\|_{H^2}\|\partial_x^{k+1} u\|_{L^2} + R_1^{(k)} + R_1^{(k+1)} \end{aligned}$$

for $0 \le k \le s-2$. We integtate this inequality with respect to t and add for k with $0 \le k \le s-2$. Noting that $\sum_{k=0}^{s-1} |E_1^{(k)}| \le C\|W\|_{H^{s-1}}^2$ and using the Young inequality, we obtain

$$\begin{aligned} \int_0^t \|v(\tau)\|_{H^{s-1}}^2 \le \varepsilon\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau + C_\varepsilon\int_0^t \|y(\tau)\|_{H^s}^2 d\tau \\ + CE_0^2 + CE(t)^2 + CE(t)D(t)^2 \end{aligned} \tag{2.16}$$

for any $\varepsilon > 0$, where C_ε is a constant depending on ε. Here we also used the following estimates for $R_1^{(k)}$:

$$R_1^{(k)} \le C\|z\|_{L^\infty}\|\partial_x^k v\|_{L^2}\|\partial_x^{k+1} z\|_{L^2}, \qquad \sum_{k=0}^{s-1}\int_0^t R_1^{(k)}(\tau)d\tau \le CE(t)D(t)^2.$$

Step 3: (Dissipation for u and z) To get the dissipation term U_x^2, we compute as (2.13a)$\times$ $(-U_x)$ + (2.13c) $\times V_x$. This gives

$$-(VU_x)_t + (VU_t)_x + U_x^2 = V_x^2 + YU_x. \tag{2.17}$$

Integrating (2.17) with respect to x, we have

$$\frac{d}{dt}E_2^{(k)} + \|\partial_x^{k+1}u\|_{L^2}^2 \le \|\partial_x^{k+1}v\|_{L^2}^2 + \|\partial_x^k y\|_{L^2}\|\partial_x^{k+1}u\|_{L^2} \tag{2.18}$$

for $0 \le k \le s-2$, where $E_2^{(k)} := -\int_{\mathbb{R}} \partial_x^k v\, \partial_x^{k+1}u\, dx$. We integrate (2.18) with respect to t and add for k with $0 \le k \le s-2$. Then we easily get

$$\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau \le C\int_0^t \|v(\tau)\|_{s-1}^2 + \|y(\tau)\|_{H^{s-2}}^2\, d\tau + CE_0^2 + CE(t)^2. \tag{2.19}$$

In order to create the dissipation term Z_x^2, we compute as $(2.13b)\times(-Z_x)+(2.13d)\times Y_x$. This yields

$$-(YZ_x)_t + (YZ_t)_x + aZ_x^2 = aY_x^2 - (V-\gamma Y)\,Z_x - Z_x\partial_x^k g(z)_x. \tag{2.20}$$

Integrating (2.20) with respect to t, we obtain

$$\frac{d}{dt}E_3^{(k)} + a\|\partial_x^{k+1}z\|_{L^2}^2 \le a\|\partial_x^{k+1}y\|_{L^2}^2 + \|\partial_x^k v - \gamma\partial_x^k y\|_{L^2}\|\partial_x^{k+1}z\|_{L^2} + R_3^{(k)} \tag{2.21}$$

for $0 \le k \le s-1$, where

$$E_3^{(k)} := -\int_{\mathbb{R}} \partial_x^k y\, \partial_x^{k+1}z\, dx, \qquad R_3^{(k)} := \int_{\mathbb{R}} |\partial_x^{k+1}z|\,|\partial_x^{k+1}g(z)|\, dx.$$

We integrate (2.21) with respect to t and add for k with $0 \le k \le s-1$. This yields

$$\begin{aligned}\int_0^t \|\partial_x z(\tau)\|_{H^{s-1}}^2 d\tau &\le C\int_0^t \|v(\tau)\|_{s-1}^2 + \|y(\tau)\|_{H^s}^2\, d\tau \\ &\quad + CE_0^2 + CE(t)^2 + CE(t)D(t)^2. \end{aligned}\tag{2.22}$$

Here we have used the estimates

$$R_3^{(k)} \le C\|z\|_{L^\infty}\|\partial_x^{k+1}z\|_{L^2}^2, \qquad \sum_{k=0}^{s-1}\int_0^t R_3^{(k)}(\tau)d\tau \le CE(t)D(t)^2.$$

Step 4: Finally, combining (2.16), (2.19) and (2.22), and then taking $\varepsilon > 0$ in (2.16) suitably small, we arrive at the estimate

$$\begin{aligned}&\int_0^t \|v(\tau)\|_{H^{s-1}}^2 + \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|\partial_x z(\tau)\|_{H^{s-1}}^2\, d\tau \\ &\le C\int_0^t \|y(\tau)\|_{H^s}^2 d\tau + CE_0^2 + CE(t)^2 + CE(t)D(t)^2.\end{aligned}$$

This combined with the basic estimate (2.11) yields the desired inequality $E(t)^2+D(t)^2 \le CE_0^2 + CE(t)D(t)^2$. Thus the proof of Proposition 2.2 is comptlete. □

3 L^2 decay estimate

The aim of this section is to show the optimal decay estimate stated in Theorem 1.3. For this purpose we derive the pointwise estimate of solutions in the Fourier space. We recall that the system (1.2) is written in the form of (2.12) or in the vector notation as

$$W_t + AW_x + LW = G_x, \tag{3.1}$$

where $G = (0, g(z), 0, 0)^T$ with $g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0)z/a = O(z^2)$ for $z \to 0$; the coefficient matrices A and L are given in (1.5).

Proposition 3.1 (Pointwise estimate). *Let W be a solution of* (3.1) *with the initial data W_0. Then the Fourier image $\hat{W}$ satisfies the pointwise estimate*

$$|\hat{W}(\xi,t)|^2 \le Ce^{-c\rho(\xi)t}|\hat{W}_0(\xi)|^2 + C\int_0^t e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{G}(\xi,\tau)|^2 d\tau \tag{3.2}$$

for $\xi \in \mathbb{R}$ and $t \ge 0$, where $\rho(\xi) := \xi^2/(1+\xi^2)^2$, and C and c are positive constants.

Our optimal decay estimate will be obtained by applying the following decay estimate of L^2-L^q-L^r type which was established in [9].

Lemma 3.2 (Decay estimate of L^2-L^q-L^r type [9]). *Let U be a function satisfying*

$$|\hat{U}(\xi,t)| \le C|\xi|^m e^{-c\rho(\xi)t}|\hat{U}_0(\xi)| \tag{3.3}$$

for $\xi \in \mathbb{R}$ and $t \ge 0$, where $\rho(\xi) = \xi^2/(1+\xi^2)^2$, $m \ge 0$, and U_0 is a given function. Then we have

$$\begin{aligned}\|\partial_x^k U(t)\|_{L^2} &\le C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+m}{2}}\|U_0\|_{L^q}\\ &\quad + C(1+t)^{-\frac{\ell}{2}+\frac{1}{2}(\frac{1}{r}-\frac{1}{2})}\|\partial_x^{k+m+\ell}U_0\|_{L^r},\end{aligned} \tag{3.4}$$

where $k \ge 0$, $1 \le q$, $r \le 2$, $\ell > \frac{1}{r} - \frac{1}{2}$ ($\ell \ge 0$ if $r = 2$).

Remark 3.1. The first (resp. the second) term on the right hand side of (3.4) is corresponding to the low frequency region $|\xi| \le 1$ (resp. high frequency region $|\xi| \ge 1$). When $m = 0$, $q = 1$ and $r = 2$, the estimate (3.4) is reduced to

$$\|\partial_x^k U(t)\|_{L^2} \le C(1+t)^{-1/4-k/2}\|U_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}U_0\|_{L^2},$$

which is just the same decay estimate obtained in [2] for the linear system (1.6).

The outline of the proof of Lemma 3.2 is as follows. From the Plancherel theorem and (3.3), we have

$$\|\partial_x^k U(t)\|_{L^2}^2 = \int_{\mathbb{R}} \xi^{2k}|\hat{U}(\xi,t)|^2 d\xi \le C\int_{\mathbb{R}} \xi^{2(k+m)}e^{-c\rho(\xi)t}|\hat{U}_0(\xi)|^2 d\xi$$

We divide the last integral into two parts corresponding to $|\xi| \le 1$ and $|\xi| \ge 1$, respectively, and estimate each part by applying the Hölder inequality and the Hausdorff-Young inequality. This yields the desired estimate (3.4). We omit the details and refer to [9].

3.1 Proof of Proposition 3.1

Taking the Fourier transform of (2.12), we have

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{3.5a}$$
$$\hat{y}_t - ai\xi\hat{z} - \hat{v} + \gamma\hat{y} = i\xi\hat{g}, \tag{3.5b}$$
$$\hat{u}_t - i\xi\hat{v} = 0, \tag{3.5c}$$
$$\hat{z}_t - ai\xi\hat{y} = 0, \tag{3.5d}$$

where $g = g(z)$. We construct a Lyapunov function of the system (3.5) in the Fourier space. The computations below are essentially the same as in [3] and correspond to those in the proof of Proposition 2.2. We divide the proof into 4 steps.

Step 1: (Basic energy and dissipation for $\hat{y}$) We compute as (3.5a) $\times\ \bar{\hat{v}}$ + (3.5b) $\times\ \bar{\hat{y}}$ + (3.5c) $\times\ \bar{\hat{u}}$ + (3.5d) $\times\ \bar{\hat{z}}$ and take the real part. This yields

$$\frac{1}{2}E_{0,t} + \gamma|\hat{y}|^2 = \mathrm{Re}\,\{i\xi\bar{\hat{y}}\hat{g}\},$$

where $E_0 := |\hat{W}|^2$. Applying the Young inequality, we have

$$E_{0,t} + \gamma|\hat{y}|^2 \le C\xi^2|\hat{g}|^2. \tag{3.6}$$

Step 2: (Dissipation for $\hat{v}$) To create the dissipation term for $\hat{v}$, we compute as (3.5b) $\times$ $(-\bar{\hat{v}})$ + (3.5a) $\times$ $(-\bar{\hat{y}})$ + (3.5c) $\times$ $(-a\bar{\hat{z}})$ + (3.5d) $\times$ $(-a\bar{\hat{u}})$ and take the real part. This gives

$$\begin{aligned} E_{1,t} + |\hat{v}|^2 - |\hat{y}|^2 &= \gamma\,\mathrm{Re}\,(\bar{\hat{v}}\hat{y}) - \mathrm{Re}\,\{i\xi(\bar{\hat{y}}\hat{u} + a^2\bar{\hat{u}}\hat{y})\} - \mathrm{Re}\,\{i\xi\bar{\hat{v}}\hat{g}\} \\ &= \gamma\,\mathrm{Re}\,(\bar{\hat{v}}\hat{y}) - (a^2-1)\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) - \xi\,\mathrm{Re}\,\{i\bar{\hat{v}}\hat{g}\}, \end{aligned}$$

where $E_1 := -\mathrm{Re}\,(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}})$. We multiply this equality by $1+\xi^2$. Then, using the Young inequality, we obtain

$$(1+\xi^2)E_{1,t} + c_1(1+\xi^2)|\hat{v}|^2 \le \varepsilon\xi^2|\hat{u}|^2 + C_\varepsilon(1+\xi^2)^2|\hat{y}|^2 + C(1+\xi^2)\xi^2|\hat{g}|^2 \tag{3.7}$$

for any $\varepsilon > 0$, where c_1 is a positive constant with $c_1 < 1$ and C_ε is a constant depending on ε.

Step 3: (Dissipation for $\hat{u}$ and $\hat{z}$) To create the dissipation term $|\hat{u}|^2$, we compute as (3.5a) $\times\ i\xi\bar{\hat{u}}$ − (3.5c) $\times\ i\xi\bar{\hat{v}}$ and take the real part. The result is

$$\xi E_{2,t} + \xi^2(|\hat{u}|^2 - |\hat{v}|^2) + \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) = 0, \tag{3.8}$$

where $E_2 := \mathrm{Re}\,(i\hat{v}\bar{\hat{u}})$. For the dissipation term $|\hat{z}|^2$, we compute as (3.5b) $\times\ i\xi\bar{\hat{z}}$ − (3.5d) $\times$ $i\xi\bar{\hat{y}}$ and take the real part. Then we have

$$\xi E_{3,t} + a\xi^2(|\hat{z}|^2 - |\hat{y}|^2) - \xi\,\mathrm{Re}\,\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\} = -\xi^2\,\mathrm{Re}\,\{\bar{\hat{z}}\hat{g}\}, \tag{3.9}$$

where $E_3 := \mathrm{Re}\,(i\hat{y}\bar{\hat{z}})$. Now we combine (3.8) and (3.9) such that (3.8) + (3.9) $\times$ $(1+\xi^2)$. This gives

$$\begin{aligned} &\xi\{E_2 + (1+\xi^2)E_3\}_t + \xi^2|\hat{u}|^2 + a(1+\xi^2)\xi^2|\hat{z}|^2 \\ &= \xi^2|\hat{v}|^2 + a(1+\xi^2)\xi^2|\hat{y}|^2 + (1+\xi^2)\xi\,\mathrm{Re}\,\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\} \\ &\quad - \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) - (1+\xi^2)\xi^2\,\mathrm{Re}\,\{\bar{\hat{z}}\hat{g}\}. \end{aligned}$$

Using the Young inequality, we get

$$\begin{aligned}&\xi\{E_2+(1+\xi^2)E_3\}_t+c_1\xi^2|\hat{u}|^2+c_2(1+\xi^2)\xi^2|\hat{z}|^2\\&\le C(1+\xi^2)|\hat{v}|^2+C(1+\xi^2)^2|\hat{y}|^2+C(1+\xi^2)\xi^2|\hat{g}|^2,\end{aligned} \tag{3.10}$$

where c_1 and c_2 are positive constants satisfying $c_1<1$ and $c_2<a$, respectively.

Step 4: (Lyapunov function) Letting $\alpha_1>0$, we combine (3.7) and (3.10) such that (3.7) + (3.10) $\times\,\alpha_1$. Then we have

$$\begin{aligned}&\big\{(1+\xi^2)E_1+\alpha_1\xi\{E_2+(1+\xi^2)E_3\}\big\}_t+(c_1-\alpha_1C)(1+\xi^2)|\hat{v}|^2\\&\quad+(\alpha_1c_1-\varepsilon)\xi^2|\hat{u}|^2+\alpha_1c_2(1+\xi^2)\xi^2|\hat{z}|^2\\&\le C_{\varepsilon,\alpha_1}(1+\xi^2)^2|\hat{y}|^2+C_{\alpha_1}(1+\xi^2)\xi^2|\hat{g}|^2,\end{aligned} \tag{3.11}$$

where C_{ε,α_1} and C_{α_1} are constants depending on (ε,α_1) and α_1, respectively. Also, letting $\alpha_2>0$, we combine (3.6) and (3.11) such that (3.6) + (3.11) $\times\,\frac{\alpha_2}{(1+\xi^2)^2}$. Then, putting

$$E:=E_0+\frac{\alpha_2}{1+\xi^2}\Big(E_1+\frac{\alpha_1\xi}{1+\xi^2}\{E_2+(1+\xi^2)E_3\}\Big), \tag{3.12}$$

we obtain

$$\begin{aligned}&E_t+\alpha_2(c_1-\alpha_1C)\frac{1}{1+\xi^2}|\hat{v}|^2+(\gamma-\alpha_2C_{\varepsilon,\alpha_1})|\hat{y}|^2\\&+\alpha_2(\alpha_1c_1-\varepsilon)\frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2+\alpha_2\alpha_1c_2\frac{\xi^2}{1+\xi^2}|\hat{z}|^2\le C_{\alpha_1,\alpha_2}\xi^2|\hat{g}|^2,\end{aligned} \tag{3.13}$$

where C_{α_1,α_2} is a constant depending on (α_1,α_2). Here we see that there is a small positive constant α_0 such that if $\alpha_1,\ \alpha_2\in(0,\alpha_0]$, then E in (3.12) is equivalent to $|\hat{W}|^2$, that is,

$$c_0|\hat{W}|^2\le E\le C_0|\hat{W}|^2, \tag{3.14}$$

where c_0 and C_0 are positive constants. Futhermore, we choose $\alpha_1\in(0,\alpha_0]$ such that $c_1-\alpha_1C>0$ and take $\varepsilon>0$ so small as $\alpha_1c_1-\varepsilon>0$. Finally, we choose $\alpha_2\in(0,\alpha_0]$ such that $\gamma-\alpha_2C_{\varepsilon,\alpha_1}>0$. Then (3.13) becomes to

$$E_t+cF\le C\xi^2|\hat{g}|^2, \tag{3.15}$$

where

$$F:=\frac{1}{1+\xi^2}|\hat{v}|^2+|\hat{y}|^2+\frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2+\frac{\xi^2}{1+\xi^2}|\hat{z}|^2 \tag{3.16}$$

This suggests that E in (3.12) is the desired Lyapunov function of the system (3.5). Noting (3.14), we find that $F\ge c\rho(\xi)E$, where $\rho(\xi)=\xi^2/(1+\xi^2)^2$. Therefore (3.15) becomes to $E_t+c\rho(\xi)E\le C\xi^2|\hat{g}|^2$. Solving this ordinary differential inequality for E and using (3.14), we arrive at the desired estimate (3.2) in the form

$$|\hat{W}(\xi,t)|^2\le Ce^{-c\rho(\xi)t}|\hat{W}_0(\xi)|^2+C\int_0^t e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{g}(\xi,\tau)|^2d\tau.$$

This completes the proof of Proposition 3.1. $\square$

3.2 Proof of Theorem 1.3

Let W be the solution to the problem (1.2), (1.3) obtained in Theorem 1.2. Then W satisfies (3.1). Therefore we have the pointwise estimate (3.2). We integrate (3.2) with respect to ξ. Applying the Plancherel theorem, we obtain

$$\begin{aligned}\|W(t)\|_{L^2}^2 &= \int_{\mathbb{R}} |\hat{W}(\xi,t)|^2 d\xi \\ &\le C\int_{\mathbb{R}} e^{-c\rho(\xi)t}|\hat{W}_0(\xi)|^2 d\xi + C\int_0^t\int_{\mathbb{R}} e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{g}(\xi,\tau)|^2 d\xi d\tau =: I + J. \quad (3.17)\end{aligned}$$

We estimate the terms I and J by applying Lemma 3.2. For I, using (3.4) with $m=0$, we have

$$\begin{aligned} I &= C\int_{\mathbb{R}} e^{-c\rho(\xi)t}|\hat{W}_0(\xi)|^2\, d\xi \\ &\le \underbrace{C(1+t)^{-\frac{1}{2}}\|W_0\|_{L^1}^2}_{k=0,\ q=1} + \underbrace{C(1+t)^{-1}\|\partial_x W_0\|_{L^2}^2}_{k=0,\ \ell=1,\ r=2} \\ &\le CE_1^2(1+t)^{-\frac{1}{2}}, \end{aligned} \quad (3.18)$$

where $E_1 = \|W_0\|_{H^2} + \|W_0\|_{L^1}$. On the other hand, for J we use (3.4) with $m=1$. Then we obtain

$$\begin{aligned} J &= C\int_0^t\int_{\mathbb{R}} e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{G}(\xi,\tau)|^2\, d\tau d\xi \\ &\le C\int_0^t \underbrace{(1+t-\tau)^{-\frac{3}{2}}\|G(\tau)\|_{L^1}^2}_{k=0,\ q=1}\, d\tau + C\int_0^t \underbrace{(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^2 G(\tau)\|_{L^1}^2}_{k=0,\ \ell=1,\ r=1}\, d\tau \\ &=: J_1 + J_2. \end{aligned}$$

Here we introduce the norms $N(t)$ and $D(t)$ by

$$N(t) = \sup_{0\le\tau\le t}(1+\tau)^{\frac{1}{4}}\|W(\tau)\|_{L^2}, \qquad D(t)^2 = \int_0^t \|\partial_x z(\tau)\|_{H^1}^2\, d\tau.$$

We know from Theorem 1.2 that $D(t) \le CE_0 \le CE_1$. For the low frequency part J_1, since $\|G\|_{L^1} \le C\|z\|_{L^2}^2$, we have

$$\begin{aligned} J_1 &\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\|z(\tau)\|_{L^2}^4\, d\tau \\ &\le CN(t)^4\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-1}\, d\tau \le CN(t)^4(1+t)^{-1}. \end{aligned} \quad (3.19)$$

For the high frequency part J_2, using $\|\partial_x^2 G\|_{L^1} \le C\|z\|_{L^2}\|\partial_x^2 z\|_{L^2}$, we have

$$\begin{aligned}
J_2 &\le C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|z(\tau)\|_{L^2}^2\|\partial_x^2 z(\tau)\|_{L^2}^2\,d\tau \\
&\le CN(t)^2\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}\|\partial_x^2 z(\tau)\|_{L^2}^2\,d\tau \\
&\le CN(t)^2 D(t)^2 \sup_{0\le\tau\le t}\left\{(1+t-\tau)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}\right\} \\
&\le CN(t)^2D(t)^2(1+t)^{-\frac{1}{2}}. \qquad (3.20)
\end{aligned}$$

Combining (3.18), (3.19) and (3.20) and using $D(t) \le CE_1$, we obtain

$$(1+t)^{\frac{1}{2}}\|W(t)\|_{L^2}^2 \le CE_1^2 + CN(t)^4 + CE_1^2N(t)^2.$$

Thus we have the inequality $N(t)^2 \le CE_1^2 + CN(t)^4 + CE_1^2N(t)^2$. This inequality can be solved as $N(t) \le CE_1$, provided that E_1 is suitably small. Thus we have proved the desired decay estimate $\|W(t)\|_{L^2} \le CE_1(1+t)^{-1/4}$. This completes the proof of Theorem 1.3. □

References

[1] K. Ide and S. Kawashima, Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system, Math. Models Meth. Appl. Sci., **18** (2008), 1001-1025.

[2] K. Ide, K. Haramoto and S. Kawashima, Decay property of regularity-loss type for dissipative Timoshenko system, Math. Models Meth. Appl. Sci., **18** (2008), 647-667.

[3] N. Mori and S. Kawashima, Decay property for the Timoshenko system with Fourier's type heat conduction, J. Hyperbolic Differential Equations, **11** (2014), 135-157.

[4] J.E. Muñoz Rivera and R. Racke, Global stability for damped Timoshenko systems, Discrete and Continuous Dynamical Systems, **9** (2003), 1625-1639.

[5] Y. Shizuta and S. Kawashima, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, Hokkaido Math. J., **14** (1985), 249-275.

[6] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, Philosophical Magazine, **41** (1921), 744-746.

[7] S.P. Timoshenko, On the transverse vibrations of bars of uniform cross-section, Philosophical Magazine, **43** (1922), 125-131.

[8] T. Umeda, S. Kawashima and Y. Shizuta, On the decay of solutions to the linearized equations of electro-magneto-fluid dynamics, Japan J. Appl. Math., **1** (1984), 435-457.

[9] J. Xu, N. Mori and S. Kawashima, L^p-L^q-L^r estimates and minimal decay regularity for compressible Euler-Maxwall equations, preprint.

Naofumi Mori
Graduate School of Mathematics, Kyushu University
Fukuoka 819-0395, Japan
n-mori@math.kyushu-u.ac.jp

Shuichi Kawashima
Faculty of Mathematics, Kyushu University
Fukuoka 819-0395, Japan
kawashim@math.kyushu-u.ac.jp

九州大学大学院 数理学府数理学専攻　森 直文